Letters

On the Expansion of Axial Field Components in Terms of Normal Modes in Perturbed Waveguides

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Abstract—Arguments are proposed according to which, in the Maxwell equations giving the axial field components, the expansion of the transverse field components in terms of unperturbed normal modes could be differentiated term by term. This would be allowed when the perturbed transverse field respects the possible nullity conditions imposed by the boundaries of the unperturbed waveguide and when it has no singularity.

I. INTRODUCTION

In various problems of perturbed waveguides the Maxwell equations can be written in the following manner:

$$\text{curl } \mathcal{E} = -i\omega\mu\mathcal{H} - \mathfrak{M} \tag{1}$$

$$\text{curl } \mathcal{H} = i\omega\epsilon\mathcal{E} + \mathcal{J}. \tag{2}$$

($\mathcal{E}$ and $\mathcal{H}$ are the unknown fields; ϵ and μ are, respectively, the permittivity and the permeability of the unperturbed waveguide; $\mathfrak{M}$ and $\mathcal{J}$ are magnetic and electric perturbations.) The terms $\mathfrak{M}$ and $\mathcal{J}$ are generally functions of $\mathcal{E}$ and $\mathcal{H}$. Note that the perturbation may appear not only in the terms $\mathfrak{M}$ and $\mathcal{J}$, but also by a modification of boundary conditions on the waveguide walls.

In order to solve (1) and (2), one usually expands the fields in terms of the normal modes of the unperturbed waveguide. In the following, we shall represent the kth of them by the components $\boldsymbol{E}_{Tk}$, E_{zk}, $\boldsymbol{H}_{Tk}$, and H_{zk}, with subscript T for transverse components and z for axial ones. Then the fundamental property to be used is the completeness of the set $\{(\boldsymbol{E}_{Tk}, \boldsymbol{H}_{Tk})\}$ on the 4-vectors defined over the cross section.[1] It ensures the existence of the expansion:

$$(\mathcal{E}_T, \mathcal{H}_T) = \sum C_k\,(\mathcal{E}_{Tk}, \mathcal{H}_{Tk}). \tag{3}$$

However, nothing general is said about axial components, and to assume 6-vector expansions of the form

$$(\mathcal{E}, \mathcal{H}) = \sum C_k\,(\mathcal{E}_k, \mathcal{H}_k) \tag{4}$$

as is done in several papers [1]–[3] has no actual justification, and often turns out to be inexact [4]. Finally, one must derive the expansion of the axial components from (3) and the Maxwell equations. For instance let us consider $\mathcal{H}_z$.

From (1), we have (with $\boldsymbol{u}_z$ as unit vector in the z direction)

$$\nabla_T\cdot(\boldsymbol{u}_z \times \mathcal{E}_T) = i\omega\mu\mathcal{H}_z + \mathfrak{M}_z. \tag{5}$$

This equation, applied to a normal mode, gives

$$\nabla_T\cdot(\boldsymbol{u}_z \times \mathcal{E}_{Tk}) = i\omega\mu H_{zk} \tag{6}$$

so that (5) can be written as

$$\nabla_T\cdot\sum C_k\,(\boldsymbol{u}_z \times \mathcal{E}_{Tk}) = i\omega\mu\,\mathcal{H}_z + \mathfrak{M}_z. \tag{7}$$

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[1] Note that as far as we know, there is no general demonstration of such completeness. The demonstration partially exposed by Collin [1] deals only with waveguides filled with homogeneous dielectrics. Bresler and Marcuvitz in Research Reports R.495.56 (May 56) and R.565.57 (Mar. 57), Microwave Research Institute, Polytechnic Institute of Brooklyn, Brooklyn, N. Y., claimed to have a general proof for closed guides filled with anisotropic and inhomogeneous dielectrics, but we do not know if they have published it. Further on, we never heard of such a proof for open waveguides.

If it is possible to take derivatives of the series $\sum C_k\boldsymbol{E}_{Tk}$ term by term, then (7) becomes

$$i\omega\mu\sum C_kH_{zk} = i\omega\mu\,\mathcal{H}_z + \mathfrak{M}_z. \tag{8}$$

When it is done, attempts to justify this assumption are generally not made [5]. However, it is not always true. As a counterexample, let us consider a perturbed waveguide with anisotropic impedance walls

$$\boldsymbol{n} \times \mathcal{E}_{tg} = \bar{Z}\cdot\mathcal{H}_{tg} \tag{9}$$

which becomes a usual empty waveguide with perfect metallic walls ($\bar{Z} = 0$) in its unperturbed state. Then we have another completeness theorem and orthogonality properties for the set $\{H_{zk}\}$ [6], which allows us to transform (5) into:

$$i\omega\mu\mathcal{H}_z + \mathfrak{M}_z = \sum_k \left[i\omega\mu\,C_k - \frac{\oint_\Gamma \boldsymbol{u}_z\cdot\bar{Z}\cdot\mathcal{H}_{tg}H_{zk}\,dl}{\iint_D H_{zk}^2\,ds} \right] H_{zk} \tag{10}$$

where D is the cross section of the guide and Γ is the contour of the cross section. This result, which does not give directly an expansion for $\mathcal{H}_z$ but an integral equation from which such an expansion may be obtained reduces to (8) only if $\bar{Z}\cdot\mathcal{H}_{tg}$ has no axial component. This is true in some important problems like the helix waveguide (with zero pitch) [5], but not in other cases such as the usual waveguides with lossy metallic walls.

II. A CRITERION

Thus we have to be cautious with the derivation of (8). In general, it is forbidden to differentiate a series term by term except under rather stringent conditions. However, the situation looks better if we admit that our series may be viewed as generalized Fourier series [7]. (They are ordinary Fourier series in the case of a slab waveguide.) Then, they may be differentiated term by term in the sense of generalized functions [8] and thus (8) will or will not be valid in the sense of true functions accordingly as the left member of (7) is itself a true function or only a generalized one. If we assume that the perturbed field has no singularity this may be deduced from the behavior of the series $\sum C_k\mathcal{E}_{Tk}$ over the cross section since only the discontinuities of its sum may produce generalized functions (Dirac functions) by differentiation.

Let this sum be $\mathcal{E}_T'$. Because of its physical nature and its continuity properties, the actual transverse field $\mathcal{E}_T$ is equal to $\mathcal{E}_T'$ everywhere [7] except on the contour of the cross section where the tangential component $\mathcal{E}_S'$ is identically zero. Thus, $\mathcal{E}_T'$ is continuous everywhere except on some lines Λ which may be either the contour of the cross section (where $\mathcal{E}_S'$ is discontinuous, unless $\mathcal{E}_S$ is also zero on the contour) or the separation lines between two different media (where the normal component $\mathcal{E}_n'$, as $\mathcal{E}_n$, is discontinuous). If we call (s, n) the local tangential and normal coordinates near a line Λ, we may write

$$\nabla_T\cdot(\boldsymbol{u}_z \times \mathcal{E}_T') = \frac{\partial\mathcal{E}_s'}{\partial n} - \frac{\partial\mathcal{E}_n'}{\partial s}. \tag{11}$$

Thus, we see that the discontinuous component is differentiated along its discontinuity direction only in the case when Λ is the contour of the cross section. Then a Dirac function appears. Nothing similar occurs in the case of separation curves, for which there is at most a step discontinuity.

Finally, the only possibility to meet a generalized function, i.e., the only case in which (8) fails, is when the tangential component of the perturbed field is not zero on the contour while the unperturbed guide has perfect metallic walls.

Now let us consider the electric analog of (8):

$$i\omega\epsilon\, \mathcal{E}_z + \mathcal{J}_z = i\omega\epsilon \sum C_k E_{zk}. \quad (12)$$

This expansion will be valid or not accordingly as

$$\nabla_T \cdot (u_z \times \mathcal{H}_T') = \frac{\partial \mathcal{H}_s'}{\partial n} - \frac{\partial \mathcal{H}_n'}{\partial s} \quad (13)$$

contains a Dirac function. This time, neither the separation lines, nor the contour have any effect, since for instance on the contour this is the normal component and not the tangential one which is identically zero with only a step discontinuity. Thus expansion (12) is always valid.

III. GENERALIZATION

In the preceding discussion we have assumed that the unperturbed waveguide has perfect metallic walls. In the case of perfect magnetic walls, we must invert our conclusions on expansions (8) and (12): (8) is always valid, while (12) fails when the perturbed transverse magnetic field has a nonzero tangential component on the contour. If in the unperturbed state, the walls have some finite, nonzero surface impedance, none of the sets $\{E_{sk}\}$ or $\{H_{sk}\}$ may be identically zero on the contour, no Dirac function may occur by the above mechanism, and the two expansions are simultaneously valid.

Let us point out that we have assumed the perturbed field has no singularity, which is sufficient for a number of practical cases. However, some perturbations such as dielectric or metallic wedges or metallic strips introduce singularities and are not covered by our theory. For instance, in the case of a microstrip line shielded in a rectangular waveguide, one may attempt to expand the field in terms of normal modes of the rectangular waveguide: then one easily finds that the expansion (12) fails.

IV. CONCLUSION

In conclusion when the perturbed field has no singularity, expansions (8) and (12) would always be valid unless the perturbed field does not respect some nullity condition imposed on the contour to the transverse tangential electric or magnetic field in the unperturbed waveguide. The possible nullity conditions on axial components would have no importance. Especially the expansions would always be valid in the case of open waveguides which have no contour. Let us recall that we have not rigorously established our proposition; we have only suggested that a suitable extension of the theory of generalized Fourier series might likely make it firmer, but such a purely mathematical work is largely beyond the scope of this letter.

REFERENCES

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[5] H. G. Unger, *Bell Syst. Tech. J.*, vol. 37, p. 1599, 1958.

[6] R. E. Collin, *Field Theory of Guided Waves.* New York: McGraw-Hill, 1960.

[7] L. Schwartz, *Méthodes Mathématiques pour les Sciences Physiques.* Paris: Hermann & Cie, 1965.

[8] V. I. Smirnov, *A Course of Higher Mathematics*, vol. 2. New York: Pergamon, 1964. (Actually the generalized Fourier series mentioned in this book ought to be further generalized in order to contain our four-vectors expansions in terms of modes.)

Correction to "Power Deposition in a Spherical Model of Man Exposed to 1–20 MHz Electromagnetic Fields"

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In the above paper,[1] Fig. 5 on page 794 should be as shown here in Fig. 1. An error occurred in translating the tabulated data into graphic form. The corrected figure is consistant with the results shown in Fig. 3 of the above paper.[1]

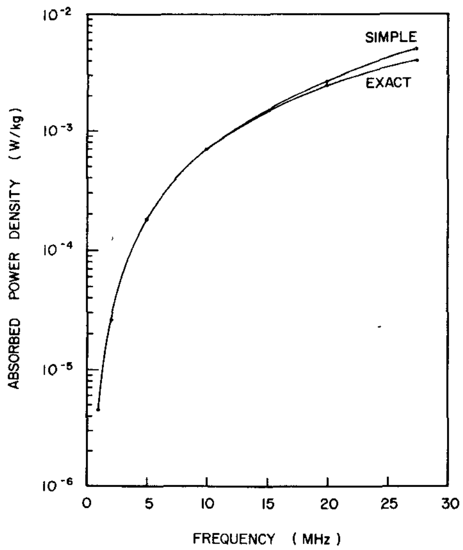

Fig. 1. Maximum absorbed power densities in a man-size sphere given by exact Mie solution and the simplified solution. Incident power density is 1 mW/cm².

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[1] J. C. Lin, A. W. Guy, and C. C. Johnson, *IEEE Trans. Microwave Theory Tech.*, vol. MTT-21, pp. 791–797, Dec. 1973.

Correction to "Variational Solution of Integral Equations"

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In the above paper,[1] two errors have been noted. First, the proof of positive-definiteness of the integral operator on page 239 requires that the potential vanish at infinity. This condition should have been stated just before (19b). By virtue of mirror image symmetries, all the examples presented satisfy this condition.

Difficulties may arise in certain two-dimensional problems because of the logarithmic Green's function. For example, if S is a circle of a radius a and the charge σ is constant:

$$\langle K\sigma, \sigma \rangle \propto -\sigma^2 \ln a.$$

Positive-definiteness holds only for $a < 1$; the form vanishes at a

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[1] B. H. McDonald, M. Friedman, and A. Wexler, *IEEE Trans. Microwave Theory Tech.*, vol. MTT-22, pp. 237–248, Mar. 1974.